from (5.5)

$$\delta = \min_{i,t} EP_i \geqslant L_{N(n)+1} = L\varkappa^{N(n)} \geqslant L\varkappa^{(2^n-1)} \quad 1 \leqslant i \leqslant n, \quad t \geqslant t_0 \tag{5.11}$$

If we select the maximum possible L allowed by inequality (5.7), then from (5.11) we obtain

$$\delta \geqslant \min\,[C_n(k)\,\varepsilon,\ C_n^*(k)\,\delta_0] \tag{5.12}$$

$$C_n(k) = {}^1/_2\, k\,(1-\varkappa)\,(e^{\lambda\pi}-1)^{-1}\varkappa^{(2^n-1)}, \qquad C_n^*(k) = \varkappa^{(2^n-1)}$$

As $\varkappa$ we can take any number from interval (5.3), for example, the $\varkappa_0$ from (5.10). We obtain explicit expressions for $\varkappa_0$, $C_n(k)$ and $C_n^*(k)$ by substituting relations (5.8), (4.9), (4.14) and (2.5) into formulas (5.10) and (5.12). In particular, when the capabilities of the pursuers approach the capability of the evading point $(k \to 1)$, we find according to the formulas indicated above

$$\varkappa_0 \approx 2\lambda e^{-4\pi\lambda}, \quad C_n(k) \approx 0.5 e^{-\pi\lambda}\varkappa_0^{(2^n-1)}$$

$$C_n^*(k) \approx \varkappa_0^{(2^n-1)}, \quad \lambda = k\,(1-k^2)^{-1/2} \to \infty, \quad k \to 1$$

We note that the evasion strategy proposed in Sect. 3 for point E, as well as the bounds (5.3) and (5.7) on the choice of parameters L and $\varkappa$, do not depend upon the number n of pursuers.

Translated by N. H. C.

UDC 62-50

ON A PROBLEM OF l-ESCAPE

PMM Vol. 40, № 1, 1976, pp. 25-37
P. B. GUSIATNIKOV
(Moscow)
(Received February 5, 1975)

We give criteria for escape and l-escape in nonlinear differential games. The paper is closely related to the investigations in [1 — 8].

1. Let the motion of a vector z in an n-dimensional Euclidean space R be described by the vector differential equation

$$dz\,/\,dt = f(t, z, u, v) \tag{1.1}$$

where $t \geqslant 0$; $u \in P$ and $v \in Q$ are control parameters varying on sets P and Q compact in R. Regarding the right-hand side of Eq. (1.1) we assume that:

a) $f(t, z, u, v)$ is continuous in $(t, z, u, v) \in X = [0, +\infty) \times R \times P \times Q$;

b) the inequality

$$|\,f(t, z_1, u, v) - f(t, z_2, u, v)\,| \leqslant k_*\,|\,z_1 - z_2\,|$$

where k_* is a constant depending only on c, is satisfied for any $u \in P, v \in Q$ and for $t \geqslant 0,\ z_1, z_2 \in R,\ |\,t\,| + |z_1| + |z_2\,| \leqslant c$;

c) a constant B exists such that

$$|\,(z \cdot f(t, z, u, v))| \leqslant B\,(1 + |z|^2)$$

holds for all $t \geqslant 0$, $z \in R$, $u \in P$, $v \in Q$;

d) the set $f(t, z, P, v)$ is convex for any $t \geqslant 0$, $z \in R$, $v \in Q$.

In addition, let a certain linear subspace M be given in R. We say that differential game (1.1) is described by all the data listed above.

The measurable vector functions $u^* = \{u(t), t \geqslant 0\}$ and $v^* = \{v(t), t \geqslant 0\}$, satisfying the inclusions $u(t) \in P$ and $v(t) \in Q$ for each t are called the controls of players U and V, respectively. The aim of player U is to lead point z onto set M; player V tries to prevent this. The game terminates when vector z first hits onto M. We note that when conditions (a)—(d) are satisfied for any $z_0 \in R$, $0 \leqslant \tau' \leqslant T$ and for any pair of controls u^* and v^* defined on $[\tau', T]$, there exists a unique [2] solution (in the sense of Carathéodory) $z(t)$, $\tau' \leqslant t \leqslant T$ of Eq. (1.1) with the initial condition $z(\tau') = z_0$ (i.e. a vector function $z(t)$, absolutely continuous on $[\tau', T]$, satisfying Eq. (1.1) almost everywhere). Function $z(t)$ is called a motion and is denoted $z(t) = z(t; \tau', z_0, u^*, v^*)$.

By π we denote the operator of orthogonal projection from R onto a subspace L (we assume that $\dim L = \nu \geqslant 2$) which is the orthogonal complement to M in R; by $\eta(t, z)$ we denote function $\eta(t, z) = (1 + t^2 + |z|^2)^{1/2}$. We set $\eta \equiv \eta(t) \equiv \eta(t, z(t))$ for any motion $z(t)$. By $D(r)$, $r \geqslant 0$ we denote the collection of all pairs (t, z) for which $\eta(t, z) \leqslant r$, by $D(r, \varepsilon)$. $r \geqslant 1$, $\varepsilon > 0$, we denote the collection of all pairs $(t, z) \in D(r)$ such that $|\pi z| \leqslant \varepsilon$ and by $X(r)$ we denote the set $D(r) \times P \times Q$.

By virtue of Eq. (1.1), for the derivative of the function η we have (see condition (c)) $|\eta\eta^{\cdot}| = |t + (z \cdot z)^{\cdot}| \leqslant |t| + B(1 + |z|^2) \leqslant (B + 1)\eta^2$, so that $|\eta^{\cdot}| \leqslant a(1 + \eta^2)$, $a = (B + 1)/2$, and, consequently [5], the estimate

$$\Phi(F(\eta_*) - \tau) = \varphi_1(\eta_*, \tau) \leqslant \eta(t) \leqslant \varphi_2(\eta_*, \tau) \equiv \Phi(F(\eta_*) + \tau), \quad 0 \leqslant \tau \leqslant \theta_* \tag{1.2}$$

$$F(r) = \frac{\operatorname{arctg} r}{a}, \quad 0 \leqslant r < +\infty; \quad \Phi(s) = \operatorname{tg} as, \quad 0 \leqslant s < \alpha = \pi/2a$$

$$\theta_* = \min\left\{\frac{\alpha - F(\eta_*)}{2}, F(1)\right\}$$

holds for any motion $z(t) = z(t; t_*, z_*, u^*, v^*)$ (here and everywhere subsequently $\tau = t - t_*$ and $\eta_* = \eta(t_*, z_*)$).

Let us consider the problem of evasion from contact (the escape problem) [1, 3—5] for differential game (1.1).

2. Let $A = A(w) \in R$ be a differentiable vector function of the vector variable $w \in R$ and let b be an arbitrary vector from R. By the product $(\partial A / \partial w \cdot b)$ we mean a vector from R, each of whose components is the scalar product of the gradient of the corresponding component of the vector function A by the vector b. We consider sequences of functions $h_i(t, z)$ and $g_i(t, z, u, v)$ satisfying the following relations:

$$\pi f(t, z, u, v) = h_1(t, z) + g_1(t, z, u, v) \tag{2.1}$$

$$\frac{\partial h_i(t, z)}{\partial t} + \left(\frac{\partial h_i(t, z)}{\partial z} \cdot f(t, z, u, v)\right) = h_{i+1}(t, z) + g_{i+1}(t, z, u, v), \quad i \geqslant 1 \tag{2.2}$$

Note that for $i = 0$ relation (2.1) can be given the form of (2.2) by setting

$$h_0(t, z) = \pi z, \quad g_0(t, z, u, v) \equiv 0 \tag{2.3}$$

We assume that the following condition is fulfilled for game (1.1) (cf [4]).

Condition 1. A positive integer k, continuously-differentiable vector functions $h_i(t, z)$, vector functions $g_i(t, z, u, v)$, $i = 1, \ldots, k$, and continuous nonnegative scalar functions $m_i(t, z, u, v)$, $i = 1, \ldots, k-1$ exist (all the functions and their properties hold on set X) such that for any $r \geqslant 1$ we can find $\gamma(r) > 0$ and $\varepsilon(r) > 0$ such that relations (2.1)—(2.3) and the inequality

$$|g_i(t, z, u, v)| \leqslant |\pi z|^{k+1-i} m_i(t, z, u, v) \tag{2.4}$$

are fulfilled for all pairs $(t, z) \in D(r, \varepsilon(r))$, for all $u \in P$, $v \in Q$ and for all $i = 0, \ldots, k-1$, and the inclusion

$$k!\,\gamma(r) S \subset \bigcap_{u \in P} g_k(t, z, u, Q), \quad (t, z) \in D(r), \quad z \in M \tag{2.5}$$

holds as well (S is the unit closed sphere in L).

Note 1. The continuity of the functions $g_i(t, z, u, v)$ on X follows from relations (2.1)—(2.3) and from the continuity of the derivatives of the functions $h_i(t, z)$. It is easy to verify as well (see the proof of Note 2) that the functions $\varepsilon(r) > 0$ and $\gamma(r) > 0$ in Condition 1 can be chosen continuous and strictly monotonically decreasing with respect to $r \geqslant 1$. Assuming that such a choice has been made, we denote the function inverse to $\varepsilon(r)$ by $E(s)$, $\varepsilon_1 < s \leqslant \varepsilon_2$ ($\varepsilon_2 = \varepsilon(1)$, $\varepsilon_1 = \lim_{r \to +\infty} \varepsilon(r)$).

Lemma 1. For every $r \geqslant 1$ and for every vector $d_0, \ldots, d_k \in L$, a vector $w \in L$, $|w| \leqslant {}^1/_2\,\gamma(r)$ exists shch that

$$\left| w\tau^k - \sum_{i=0}^{k} d_i \tau^i \right| \geqslant 4\rho(r)\tau^k, \quad 0 \leqslant \tau \leqslant 1 \tag{2.6}$$

$$\rho(r) = \min\{1, \gamma(r) / (128(k+2)^2)\}$$

Lemma 1 is a direct corollary of assertions (A) and (B) in Sect. 4 of [1].

For a fixed $r \geqslant 1$ we denote the modulus of continuity of the function $g_k(t, z, u, v)$ on set $X(r)$ by $\omega(r; \delta)$. Obviously, $\omega(r_1; \delta_1) \leqslant \omega(r_2; \delta_2)$ for $r_1 \leqslant r_2$ and $\delta_1 \leqslant \delta_2$. We set

$$\lambda(r) = \sup\left\{\left|\frac{\partial h_k(t, z)}{\partial t} + \left(\frac{\partial h_k(t, z)}{\partial z} \cdot f(t, z, u, v)\right)\right| + |f(t, z, u, v)| + \sum_{i=1}^{k-1} m_i(t, z, u, v)\right\} + r \tag{2.7}$$

(the sup here is taken over the set $X(r)$)

$$H(r) = \lambda(\Phi({}^1/_2\,(\alpha + F(r)))) + 1, \quad \mu(r; \delta) = \omega(H(r); \delta) \tag{2.8}$$

The functions $\lambda(r)$ and $H(r)$ are continuous and increase strictly monotonically in $r \geqslant 1$. Let $\Delta(r)$, $r \geqslant 1$ be such that $\Delta(r) > 0$ and

$$\mu(r; \Delta(r)) \leqslant \rho(r), \quad r \geqslant 1 \tag{2.9}$$

Note 2. The function $\Delta(r)$ satisfying (2.9) can be chosen (from now on we take this choice as made) continuous and decreasing strictly monotonically in $r \geqslant 1$.

In fact, let $\Delta_n > 0$, $n = 1, 2, \ldots$ be such that $\mu(n; \Delta_n) \leqslant \rho(n)$. Setting $\delta_n = \min\{\Delta_1, \ldots, \Delta_n\} / 2^n$, $n = 1, 2, \ldots$, we have $\delta_n \leqslant \Delta_n$ and the sequence of $\delta_n > 0$ decreases strictly monotonically. Setting

$$\Delta(r) = \delta_{n+1} + (r-n)\ (\delta_{n+2} - \delta_{n+1}), \quad n \leqslant r \leqslant n+1,\ n = 1, 2, \dots$$

we have $\Delta(r) \leqslant \delta_{n+1}$, $n \leqslant r \leqslant n+1$, so that

$$\mu(r;\ \Delta(r)) \leqslant \mu(n+1;\ \delta_{n+1}) \leqslant \rho(n+1) \leqslant \rho(r)$$

(we have used the monotonic decreasing of $\rho(r)$). We set

$$4b(r) = \min\{\varepsilon(r), \Delta(r), \rho(r), 1\},\ c(r) = \min_{1 \leqslant s \leqslant r}\left(F\left(E\left(\frac{\varepsilon(s)+\varepsilon_1}{2}\right)\right) - F(s)\right) \tag{2.10}$$

$$\theta(r) = \min\{F(1), {}^1/_2\,(\alpha - F(r)), b(r)/H(r), c(r), \rho(r)/(3^{k+1} \times H^{k+2}(r))\}$$

$$n_1(r) = b(r)\,\theta^k(r) \leqslant b(r), \quad n(r) = n_1(\Phi({}^1/_3\,(2F(r) + \alpha))) \leqslant n_1(r)$$

The functions $\theta(r)$, $n(r)$ and $b(r)$ decrease monotonically in $r \geqslant 1$ and are positive and continuous. By N we denote the collection of all pairs (t, z), $t \geqslant 0$, $z \in R$, such that

$$|\pi z| \leqslant n(\eta(t, z)) \tag{2.11}$$

3. Lemma 2. For every $(t_*, z_*) \in N$ the inequality $|\pi z(s)| \leqslant \varepsilon(\eta(s))$, $s \in I_*$ is satisfied for any motion $z(t) = z(t;\ t_*, z_*, u^*, v^*)$ on the interval $I_* = [t_*, t_* + \theta(\eta_*)]$ (we retain this notation in what follows).

In fact (here and later $f(s) \equiv f(s, z(s), u(s), v(s))$ and $g_i(s) \equiv g_i(s, z(s), u(s), v(s))$, $i = 1, \dots, k$)

$$z(t) - z_* = \int_{t_*}^{t} f(s)\,ds, \quad \pi z(t) = \pi z_* + \int_{t_*}^{t} \pi f(s)\,ds \tag{3.1}$$

So that by virtue of (2.7),(2.8) and (1.2)

$$|\pi z(t)| \leqslant |\pi z_*| + \int_{t_*}^{t} \lambda(\eta(s))\,ds \leqslant |\pi z_*| + \lambda(\Phi(F(\eta_*) + \tau))\tau \leqslant |\pi z_*| + \tau H(\eta_*), \quad t \in I_* \tag{3.2}$$

$$|z(t) - z_*| \leqslant \lambda(\varphi_2(\eta_*, \tau))\tau \leqslant \tau(H(\eta_*) - 1), \quad t \in I_*$$

From the first inequality in (3.2) we have (see (2.10) and (2.11))

$$|\pi z(t)| \leqslant {}^1/_4\,\varepsilon(\eta_*) + b(\eta_*) \leqslant {}^1/_2\,(\varepsilon(\eta_*) + \varepsilon_1) \leqslant \varepsilon(\varphi_2(\eta_*, \tau)) \leqslant \varepsilon(\eta(t)), \quad t \in I_*$$

The last inequality follows from the monotony of $\varepsilon(r)$ and from inequality (1.2),while the penultimate one is a consequence of the inequality

$$\tau + F(\eta_*) \leqslant F(E({}^1/_2\,(\varepsilon(\eta_*) + \varepsilon_1)))$$

following from (2.10) as well as of the monotony of the functions $\Phi(s)$ and $\varepsilon(r)$. The lemma is proved.

Lemma 3. When Condition 1 is satisfied, the equality

$$\pi z(t) = T(\tau) + m(t) + I(t) + h(t) \tag{3.3}$$

holds for any $0 \leqslant t_* \leqslant t$ and for any motion $z(t) = z(t;\ t_*,\ z_*,\ u^*,\ v^*)$ such that $|\pi z(s)| \leqslant \varepsilon(\eta(s)),\ t_* \leqslant s \leqslant t$. Here

$$I(t) = \int_{t_*}^{t} \frac{(t-s)^{k-1}}{(k-1)!} g_k(t_*, z_* - \pi z_*, u(s), v(s))\, ds, \tag{3.4}$$

$$m(t) = \sum_{i=1}^{k-1} \int_{t_*}^{t} \frac{(t-s)^{i-1}}{(i-1)!} g_i(s)\, ds$$

$$T(\tau) = \pi z_* + \sum_{i=1}^{k} \frac{h_i(t_*, z_*)}{i!} \tau^i, \quad h(t) = \int_{t_*}^{t} \frac{(t-s)^{k-1}}{(k-1)!} p(s)\, ds \tag{3.5}$$

$$p(s) = \frac{t-s}{k}\left[\frac{\partial h_k(s, z(s))}{\partial t} + \left(\frac{\partial h_k(s, z(s))}{\partial z} \cdot f(s)\right)\right] + \tag{3.6}$$
$$[g_k(s) - g_k(t_*, z_* - \pi z_*, u(s), v(s))]$$

Lemma 3 follows from the second equality in (3. 1) by a k-fold integration by parts with due regard to relations (2. 1) — (2. 3).

Now let $(t_*,\ z_*) \in N,\ t \in I_*$ and $z(t) = z(t;\ t_*,\ z_*,\ u^*,\ v^*)$ be an arbitrary motion (we recall that by notation $z(t_*) = z_*$). From Lemma 2 it follows that Lemma 3, together with (3. 3) — (3. 6), is valid for $z(t)$. Let us estimate the second and the fourth terms in (3. 3). By virtue of (2. 4),(2. 7), (2. 8),(1. 2) and (3. 2) we have

$$|m(t)| \leqslant \sum_{i=1}^{k-1} \int_{t_*}^{t} \frac{(t-s)^{i-1}}{(i-1)!} |\pi z(s)|^{k+1-i} m_i(s)\, ds \leqslant \tag{3.7}$$
$$H(\eta_*) \sum_{i=1}^{k-1} \tau^i (|\pi z_*| + \tau H(\eta_*))^{k+1-i}, \quad t \in I_*$$

(here $m_i(s) \equiv m_i(s,\ z(s),\ u(s),\ v(s)),\ i = 1, \ldots,\ k-1$). For $p(s)$ we have the estimate

$$|p(s)| \leqslant \tau\lambda(\eta(s)) + \omega(H(\eta_*); \tau + |z(s) - z_*| + |\pi z_*|) \tag{3.8}$$

(see (2.8) and the properties of $\omega(r;\ \delta)$), because

$$\eta(t_*,\ z_* - \pi z_*) \leqslant \eta_* \leqslant H(\eta_*), \quad \eta(s) \leqslant \Phi(^1/_2(F(\eta_*) + \alpha)) \leqslant H(\eta_*)$$

$$[(s - t_*)^2 + |z(s) - (z_* - \pi z_*)|^2]^{1/2} \leqslant \tau + |z(s) - z_*| + \pi z_*|$$

From (3. 1) follows (see (3. 2)) $\tau + |z(s) - z_*| \leqslant \tau H(\eta_*)$; therefore, finally (see (2. 9) and (2. 10)):

$$|h(t)| \leqslant \tau^{k+1} H(\eta_*) + \tau^k \mu(\eta_*), \ |\pi z_*| + \tau H(\eta_*) \leqslant \tau^k (\tau H(\eta_*) + \rho(\eta_*)), \quad t \in I_* \tag{3.9}$$

(here we have used the inequalities $|\pi z_*| \leqslant {}^1/_4\, \Delta(\eta_*)$ and $\tau H(\eta_*) \leqslant \theta(\eta_*) \times H(\eta_*) \leqslant {}^1/_4\, \Delta(\eta_*)$).

4. We introduce the notation of a special control of player V. Let $(t_*, z_*) \in N$ and let $w \in {}^1/_2\, \gamma(\eta_*) S$. Then, for every control $u^* = \{u(s), s \geqslant t_*\}$ a control $v_w^* = \{v_w(s) = V(u(s), w)\}$, called special, exists (we drop the indices showing dependency on t_* and z_*) such that

$$g_k(t_*,\ z_* - \pi z_*,\ u(s),\ v_w(s)) = -k!w,\quad s \geqslant t_* \tag{4.1}$$

The function $V(u, w)$ can be chosen [3] so that the function $v_w(s)$, $s \geqslant t_*$ is measurable for any measurable u^* and for fixed w. Multiplying (4.1) by $(t-s)^{k-1}$ and integrating from t_* to t, we obtain

$$I(t) = \int_{t_*}^{t} \frac{(t-s)^{k-1}}{(k-1)!} g_k(t_*, z_* - \pi z_*, u(s), v_w(s))\, ds = -w\tau^k,\quad \tau \geqslant 0 \tag{4.2}$$

5. Let us describe the active behavior of player V. Let game (1.1) commence at the point $(t_*, z_*) \in N$. Then, by virtue of Lemma 1, we can find vector

$$w = w(t_*,\ z_*) \in {}^1/_2\, \gamma(\eta_*)\, S$$

for the polynomial $T(\tau)$ given by formula (3.5), such that $|T(\tau) - w\tau^k| \geqslant 4\rho(\eta_*)\tau^k$, $0 \leqslant \tau \leqslant \theta(\eta_*)$. We fix the vector w thus found and we direct player V to apply the special control $v(s) \equiv v_w(s)$ (see Sect. 4) on interval I_*. Then from Lemma 3, the estimate (3.7) and (3.9) and the equalities (3.3) – (3.6) and (4.2) we have

$$|\pi z(t)| \geqslant 4\rho(\eta_*)\tau^k - \sum_{i=1}^{k-1} \tau^i H(\eta_*)(|\pi z_*| + \tau H(\eta_*))^{k+1-i} - \tau^k(\tau H(\eta_*) + \rho(\eta_*)) \tag{5.1}$$

for $t \in I_*$.

From formula (3.1) follows (see (3.2))

$$|\pi z(t)| \geqslant |\pi z_*| - \tau H(\eta_*),\quad t \in I_*$$

Therefore, for all $\tau \in [0,\ \theta(\eta_*)] \cap [0,\ |\pi z_*| / (2H(\eta_*))]$ we have

$$|\pi z(t)| \geqslant {}^1/_2 |\pi z_*| \geqslant {}^1/_2 |\pi z_*|^k \tag{5.2}$$

Formula (5.1) yields

$$|\pi z(t)| \geqslant 3\rho(\eta_*)\tau^k - \tau^{k+1}H(\eta_*)\Big[1 + \sum_{i=1}^{k-1}\Big(\frac{|\pi z_*|}{\tau} + H(\eta_*)\Big)^{k+1-i}\Big] \geqslant \tag{5.3}$$

$$3\rho(\eta_*)\tau^k - \tau^{k+1}H(\eta_*)\Big[1 + \sum_{i=1}^{k-1}(3H(\eta_*))^{k+1-i}\Big] \geqslant$$

$$\tau^k[3\rho(\eta_*) - \tau 3^{k+1}(H(\eta_*))^{k+2}] \geqslant 2\rho(\eta_*)\tau^k > \rho(\eta_*)\frac{|\pi z_*|^k}{(2H(\eta_*))^k}$$

for those same $\tau \in [0,\ \theta(\eta_*)]$ for which $\tau > |\pi z_*| / (2H(\eta_*))$ (see (2.10)). Since by virtue of (1.2)

$$F(\eta(t)) \geqslant F(\eta_*) - \tau \geqslant F(\eta_*) - \frac{\alpha - F(\eta_*)}{2} \geqslant \frac{3F(\eta_*) - \alpha}{2},\quad t \in I_*$$

and, consequently,

$$F(\eta_*) \leqslant \frac{2F(\eta(t)) + \alpha}{3},\quad t \in I_* \tag{5.4}$$

then

$$|\pi z(t)| \geqslant q(\eta(t))\, |\pi z_*|^k,\quad t \in I_* \tag{5.5}$$

$$q(r) = \min\Big\{\frac{1}{2}, \rho\Big(\Phi\Big(\frac{2F(r)+\alpha}{3}\Big)\Big) \Big/ [2H(\Phi(\cdot))]^k\Big\} \tag{5.6}$$

together with (5.2) and (5.3) yield the monotony of the functions $\rho(r)$ and $H(r)$. According to the penultimate inequality in (5.3), we have (see (5.4))

$$|\pi z(t^*)| \geqslant 2\rho(\eta_*)\ \ (\theta(\eta_*))^k > n(\eta(t^*)) \tag{5.7}$$

at the instant $t^* = t_* + \theta(\eta_*)$ (i.e. when $\tau = \theta(\eta_*)$) ; therefore, the point $(t^*, z(t^*))$ does not belong to N.

6. Theorem on evasion from contact. Evasion from contact is possible if Condition 1 is fulfilled for the differential game (1.1). Here, for every initial point (t_0, z_0) of the game, $t_0 \geqslant 0$, $z_0 \in R \setminus M$, a suitable choice of the escape control $v^* = \{v(t),\ t \geqslant t_0\}$ can ensure the following estimate for the quantity $\xi(t) = |\pi z(t)|,\ \ t \geqslant t_0$:

$$\xi(t) \geqslant \begin{cases} \delta(\eta(t)), & t \geqslant t_0 & \text{for } (t_0, z_0) \notin N \\ (\xi(t_0))^k \delta(\eta(t)), & 0 \leqslant t - t_0 \leqslant \theta(\eta(t_0, z_0)) & \text{for } (t_0, z_0) \in N \\ \delta(\eta(t)), & t \geqslant t_0 + \theta(\eta(t_0, z_0)) & \text{for } (t_0, z_0) \in N \end{cases} \tag{6.1}$$

Here $\delta(r)$ and $\theta(r)$, $r \geqslant 1$ are monotonically-decreasing continuous positive functions of their argument, depending only on game (1.1) and not depending either on the initial values of the players' phase coordinates or on the run of the game, N is a certain fixed closed domain, depending only on game (1.1), in the space $[0, +\infty) \times R$, whose interior contains the set $[0, +\infty) \times M$.

Proof. We fix set N by formula (2.11). Then, no matter how player U constructs his own control $u^* = \{u(t),\ t \geqslant t_0\}$, we propose that player V conducts the escape (recall that [1] at each instant t player V knows $z(s)$ and $u(s)$, $s \leqslant t$) inductively by cycles so that each m-th cycle $(m \geqslant 1)$ consists of two intervals: (1) the interval of passive escape of duration τ_m, on which player V applies the control $v^* = v_0^* = \{v(t) \equiv v_0\}$, where once and forever v_0 is a fixed vector from Q, and (2) the interval of active escape, following it, of duration θ_m, on which player V applies the special escape control $\overset{*}{v}_{w_m}$ (see Sect. 5 for the choice of w_m). The duration of each interval is determined inductively in the following way: $\tau_1 = 0$ if $(t_0, z_0) \in N$ and $\tau_1 > 0$ is the smallest positive root of the equation $|\pi z(t_0 + \tau_1)| = n(\eta(t_0 + \tau_1))$, where $z(t) \equiv z(t;\ t_0,\ z_0,\ u^*,\ v_0^*)$ if $(t_0, z_0) \notin N$; $\theta_1 = \theta(\eta(t_0 + \tau_1))$ see (2.10)); here $w_1 \equiv w(t_0 + \tau_1,\ z(t_0 + \tau_1))$ on the first active part.

For $m \geqslant 2$ the quantity $\tau_m > 0$ is the smallest positive root (see (5.7)) of the equation

$$|\pi z(T_{m-1} + \tau_m)| = n(\eta(T_{m-1} + \tau_m))$$

where

$$T_i = t_0 + \sum_{j=1}^{i} (\tau_j + \theta_j), \quad z(t) \equiv z(t; T_{m-1}, z(T_{m-1}), u^*, v_0^*)$$

The quantity $\theta_m > 0$ is given by the formula $\theta_m = \theta(\eta(\overset{*}{T}_{m-1}))$, where $\overset{*}{T}_{m-1} = T_{m-1} + \tau$; $w_m \equiv w(\overset{*}{T}_{m-1},\ z(\overset{*}{T}_{m-1}))$ and $z(t) \equiv z(t;\ \overset{*}{T}_{m-1},\ z(\overset{*}{T}_{m-1}),\ u^*,\ \overset{*}{v}_{w_m})$ on the m-th active part.

Let us obtain estimate (6.1). We set

$$\delta(r) = \min\left\{n(r),\quad q(r)\, n^k\left(\Phi\left(\frac{2F(r)+\alpha}{3}\right)\right)\right\} \tag{6.2}$$

Then on the passive part (see (2.11) and (5.7)) the estimate follows from the definition of set N

$$\xi(t) \geqslant n(\eta(t)) \geqslant \delta(\eta(t)) \tag{6.3}$$

On the m-th active part $(m \geqslant 2)$, by virtue of (5.5), (5.4) and (6.2) we have

$$\xi(t) \geqslant [\xi(T_{m-1}^{*})]^{k} q(\eta(t)) = n^{k}(\eta(T_{m-1}^{*})) q(\eta(t)) \geqslant \delta(\eta(t)) \quad (6.4)$$

On the first active part the estimate (6.1) follows from (5.5) if $\tau_1 = 0$ and coincides with (6.4) if $\tau_1 > 0$.

Let us now show that $T_m \to +\infty$ as $m \to +\infty$. We assume the contrary. If $T_m \to T_0$, then the series $\theta_1 + \theta_2 + \ldots \leqslant T_0 - t_0$ converges, so that $\theta_j \to 0$, as $j \to +\infty$. Hence, by virtue of the monotony of the function $\theta(r) > 0$ we have $\eta(T_{m-1}^{*}) \to +\infty$. Since $T_{m-1}^{*} \to T_0$, it follows that $|z(T_{m-1}^{*})| \to +\infty$ as $T_{m-1}^{*} \to T_0$. However, the latter contradicts condition (c) of Sect. 1. The theorem is proved.

7. Condition 2. The numbers $\tau_0 \geqslant 0$ and $\theta > 0$ and the function $r(t)$, continuous on $[0, 2\theta]$ and positive on $(0, 2\theta]$, depending only on game (1.1), exist such that for any $t_0 \geqslant \tau_0$ and $z_0 \in R$ we can construct a Volterra operator [1] $S_t = S_t(t_0, z_0, u^*)$ defined on the interval $I(t_0) = [t_0 + 2\theta]$ and associating with the initial value (t_0, z_0) and with the control $u^* = \{u(t),\ t \in I(t_0)\}$ the control

$$\bar{v}^* = \{v(t) \equiv S_t(t_0, z_0, u^*),\ t \in I(t_0)\} \quad (7.1)$$

such that for every control u^* the inequality

$$|\pi z(t)| \geqslant r(t - t_0), \quad t \in I(t_0) \quad (7.2)$$

is fulfilled for the motion $z(t) = z(t;\ t_0, z_0, u^*, \bar{v}^*)$ ($\bar{v}^*$ is given by formula (7.1)).

Condition 3. Constants $K > 0$ and $D > 0$ exist such that

$$|f(t, z_1, u, v_1) - f(t, z_2, u, v_2)| \leqslant K|z_1 - z_2| + D \quad (7.3)$$

for all $t \geqslant \tau_0$, $z_1, z_2 \in R$, $u \in P$, $v_1, v_2 \in Q$.

Theorem on l-escape. Let Conditions 2 and 3 be satisfied for game (1.1). Then, for every $t_0 \geqslant \tau_0$ and $z_0 \in R$, l-evasion from contact [6] is possible for a game starting from the point (t_0, z_0), where

$$l = \min_{s \in [0, \theta]} \mu(s) > 0, \quad \mu(s) = \max\{r(s), \rho(s)\}, \quad \rho(s) = r(\theta + s) - D\frac{e^{Ks} - 1}{K}$$

Proof. Let $t_0 \geqslant \tau_0$ and $z_0 = z(t_0)$. We set $\beta_n = t_0 + n\theta$, $n = 0, 1, \ldots$. For an escape beginning at instant t_0 from point z_0 we propose to construct inductively on each of the intervals $I_n = [\beta_n, \beta_{n+1})$, $n = 0, 1, 2, \ldots$ the control $v = v(t)$ by the rule

$$v_n^* = \{v_n(t) \equiv S_t(\beta_n, z_n, u_n^*),\ t \in I_n\},\ u_n^* = \{u(t),\ t \in I_n\} \quad (7.4)$$

where $z_n = z(\beta_n)$ is the value of vector $z(t)$ at instant β_n. Then, the inequality

$$|\pi z(t)| = |\pi z(t;\ \beta_n, z_n, u_n^*, v_n^*)| \geqslant r(t - \beta_n), \quad \beta_n \leqslant t \leqslant \beta_{n+1}$$

is valid by virtue of (7.2). For $n \geqslant 1$ we have the representation

$$\pi z(t) = \pi z_n(t) + y_n(t), \quad t \in I_n$$

$$z_n(t) \equiv z(t;\ \beta_{n-1}, z_{n-1}, u_{n0}^*, v_{n0}^*), \quad u_{n0}^* = \{u(t),\ \beta_{n-1} \leqslant t \leqslant \beta_{n+1}\}$$

$$v_{n0}^* = \{v_{n0}(t) \equiv S_t(\beta_{n-1}, z_{n-1}, u_{n0}^*),\ \beta_{n-1} \leqslant t \leqslant \beta_{n+1}\}$$

$$x_n(t) = z(t;\ \beta_n, z_n, u_n^*, v_n^*) - z(t;\ \beta_{n-1}, z_{n-1}, u_{n0}^*, v_{n0}^*)$$

$$y_n(t) = \pi x_n(t)$$

Since $x_n(\beta_n) = 0$, by virtue of (3.1) and (7.3)

$$|z(t) - z_n(t)| = |x_n(t)| = \left| \int_{\beta_n}^{t} (f(s, z(s), u(s), v_n(s)) - f(s, z_n(s), u(s) \right.$$

$$\left. v_{n0}(s))) ds \right| \leqslant \int_{\beta_n}^{t} (K|z(s) - z_n(s)| + D) ds$$

Hence, by virtue of the Gronwall inequality [7]

$$|x_n(t)| \leqslant D(e^{K(t-\beta_n)} - 1)/K = h(t - \beta_n), \quad t \in I_n$$

So that $|y_n(t)| \leqslant |x_n(t)| \leqslant h(t - \beta_n)$ and, hence (see Condition 2)

$$|\pi z(t)| \geqslant |\pi z_n(t)| - |y_n(t)| \geqslant r(t - \beta_{n-1}) - h(t - \beta_n) = \rho(t - \beta_n), \quad t \in I_n \tag{7.5}$$

The theorem's assertion follows from formulas (7.4) and (7.5). In conclusion we merely note that since $r(s) > 0$, $0 < s \leqslant \theta$ and $\rho(0) = r(\theta) > 0$, we have $\mu(s) > 0$, $0 \leqslant s \leqslant \theta$, while by virtue of the continuity of $\mu(s)$ the last inequality implies the positiveness of l.

Pontriagin [1] and Nikol'skii [8] showed that Conditions 2 and 3 are fulfilled when their escape conditions are fulfilled.

8. Let us consider a problem with a small parameter. We assume additionally that the right-hand side of Eq. (1.1) can be represented as

$$f(t, z, u, v) = F(t, z, u, v) + \varepsilon\varphi(t, z, u, v) \tag{8.1}$$

where the function $F(t, z, u, v)$ satisfies conditions (a) — (c) of Sect. 1 and $\varepsilon \in [0, 1]$ is a nonnegative parameter. We introduce the dependence of the right-hand side of (1.1) on parameter ε into the notation. For a given $\varepsilon \in [0, 1]$ we denote the game (1.1) by $(1.1)_\varepsilon$, and the motions of this game by $z(t; \varepsilon) \equiv z(t; t_0, z_0, u^*, v^*, \varepsilon)$.

Condition 4. When $\varepsilon = 0$ the game $(1.1)_0$ satisfies Conditions 2 and 3, and in Condition 3

$$|f(t, z_1, u, v) - f(t, z_2, u, v)| \equiv |F(t, z_1, u, v) - F(t, z_2, u, v)| \leqslant K|z_1 - z_2| \tag{8.2}$$

for all $t \geqslant \tau_0$, $z_1, z_2 \in R$, $u \in P$, $v \in Q$.

Condition 5. The function $\varphi(t, z, u, v)$ is uniformly bounded on $[\tau_0, +\infty) \times R \times P \times Q$; namely

$$|\varphi(t, z, u, v)| \leqslant 1 \tag{8.3}$$

Theorem on escape in a small-parameter problem. Let Conditions 4 and 5 be satisfied for game (1.1). Then for any initial point (t_*, z_*) of game (1.1), at which $t_* \geqslant \tau_0$ and $z_* \notin M$, there exists $\varepsilon_* = \varepsilon(t_*, z_*) > 0$ such that evasion from contact is possible in the game $(1.1)_\varepsilon$ with initial condition $z(t_*; \varepsilon) = z_*$ for every $\varepsilon \in [0, \varepsilon_*]$.

Proof. We set

$$r_0 = \min_{\theta\leqslant s\leqslant 2\theta} r(s) > 0, \quad \Gamma_1(r) = \sup_{X(r)} |F(t, z, u, v)| + 1$$
$$\Gamma(r) = \Gamma_1(re^{(B+1)})$$

We fix (see θ in Condition 2)

$$\tau_* = \min\left\{1,\ \theta,\ \frac{|\pi z_*|}{2\Gamma(\eta_*)},\ \frac{1}{K}\ln\left(1 + \frac{Kr_0}{2D}\right)\right\} \tag{8.4}$$
$$r_* = \min_{s\in[\tau_*,\ \theta]} r(s) > 0;\ \ \varepsilon_* = \varepsilon(t_*, z_*) = \min\left\{\frac{r_*K}{2(e^{K\theta}-1)},\ \frac{r_0K}{4(e^{2K\theta}-1)}\right\}$$

For $\varepsilon \in [0, 1]$ from (3.1) (as in (3.2)) we have (by virtue of the inequality $|\eta^\cdot| \leqslant (B + 1)\eta$ obtained in Sect. 1) the inequality

$$|\pi z(t; \varepsilon)| \geqslant |\pi z_*| - \tau\Gamma(\eta_*), \quad 0 \leqslant \tau = t - t_* \leqslant 1$$

for any motion $z(t; \varepsilon) \equiv z(t;\ t_*,\ z_*,\ u^*,\ v^*;\ \varepsilon)$. So that

$$|\pi z(t;\ \varepsilon)| \geqslant {}^1/_2 |\pi z_*|, \quad 0 \leqslant \tau \leqslant \tau_* \tag{8.5}$$

Now let ε be an arbitrary fixed number from the segment $[0,\ \varepsilon_*]$. We set $\beta_n = t_* + n\theta,\ n = 0,\ 1,\ 2, \ldots$ For an escape in game $(1.1)_\varepsilon$, starting at instant t_* from point z_*, we propose to construct inductively on each of the intervals $I_n = [\beta_n,\ \beta_{n+1})$, $n = 0, 1, \ldots$ the control $v = v(t)$ by the rule

$$v_n^* \equiv \{v_n(t) \equiv S_t(\beta_n,\ z_n,\ u_n^*),\ t \in I_n\} \tag{8.6}$$
$$z_n = z(\beta_n;\ \varepsilon),\ u_n^* = \{u(t),\ t \in I_n\}$$

where operator S_t is constructed (see Condition 2) for the game $(1.1)_0$.

For $n \geqslant 1$ we have the representation

$$\pi z(t;\ \varepsilon) = \pi z_n(t) + x_n(t) + y_n(t), \quad t \in I_n \tag{8.7}$$
$$z_n(t) \equiv z(t;\ \beta_{n-1},\ z_{n-1},\ u_{n0}^*,\ v_{n0}^*;\ 0), \quad u_{n0}^* = \{u(t),\ \beta_{n-1} \leqslant t \leqslant \beta_{n+1}\}$$
$$v_{n0}^* = \{v_{n0}(t) \equiv S_t(\beta_{n-1},\ z_{n-1},\ u_{n0}^*),\ \beta_{n-1} \leqslant t \leqslant \beta_{n+1}\}$$
$$x_n(t) = \pi\varepsilon_n(t), \quad y_n(t) = \pi\Delta_n(t), \quad \Delta_n(t) = z_n(t;\ 0) - z_n(t)$$
$$\varepsilon_n(t) = z(t;\ \beta_n,\ z_n,\ u_n^*,\ v_n^*;\ \varepsilon) - z_n(t;\ 0) \equiv z(t;\ \varepsilon) - z_n(t;\ 0)$$
$$z_n(t;\ 0) \equiv z(t;\ \beta_n,\ z_n,\ u_n^*,\ v_n^*;\ 0)$$

Let us estimate each of the summands in (8.7). By virtue of condition (7.2)

$$|\pi z_n(t)| \geqslant r(t - \beta_{n-1}), \quad t \in I_n, \quad n \geqslant 1 \tag{8.8}$$

Since $\varepsilon_n(\beta_n) = 0$,

$$\varepsilon_n(t) = \int_{\beta_n}^{t} \frac{d\varepsilon_n(s)}{ds}\, ds = \int_{\beta_n}^{t} (f(s,\ z(s,\ \varepsilon),\ u(s),\ v_n(s)) - F(s,\ z_n(s;\ 0)$$
$$u(s),\ v_n(s)))\, ds$$

So that

$$|\varepsilon_n(t)| \leqslant \varepsilon \int_{\beta_n}^{t} |\varphi(s, z(s;\varepsilon), u(s), v_n(s))|\, ds + \int_{\beta_n}^{t} |F(s, z(s;\varepsilon),$$
$$u(s), v_n(s)) - F(s, z_n(s;0), u(s), v_n(s))|\, ds$$

Hence, by virtue of Conditions 4 and 5

$$|\varepsilon_n(t)| \leqslant (t-\beta_n)\varepsilon + \int_{\beta_n}^{t} K|\varepsilon_n(s)|\, ds$$

So that by Gronwall's lemma

$$|x_n(t)| \leqslant |\varepsilon_n(t)| \leqslant \varepsilon c(t-\beta_n), \quad t \in I_n, \quad n = 0, 1, \ldots; \tag{8.9}$$
$$c(s) = (e^{KS} - 1) / K$$

For the quantity $\Delta_n(t)$ we have $\Delta_n(\beta_n) = z(\beta_n; \varepsilon) - z_{n-1}(\beta_n; 0)$. Hence by virtue of (8.9)

$$|\Delta_n(\beta_n)| < \varepsilon c, \quad c = c(\theta), \quad n = 1, 2, \ldots \tag{8.10}$$

Therefore, according to Condition 3

$$|\Delta_n(t)| \leqslant \varepsilon c + \int_{\beta_n}^{t} |F(s, z_n(s;0), u(s), v_n(s)) - F(s, z_n(s), u(s)$$
$$v_{n0}(s))|\, ds \leqslant \varepsilon c + \int_{\beta_n}^{t} (K|\Delta_n(s)| + D)\, ds$$

Hence by Gronwall's lemma

$$|y_n(t)| \leqslant |\Delta_n(t)| \leqslant \left(\varepsilon c + \frac{D}{K}\right) e^{K(t-\beta_n)} - \frac{D}{K} \leqslant \varepsilon c e^{K\theta} + Dc(t-\beta_n) \quad t \in I_n \tag{8.11}$$

Combining (8.7)—(8.9) and (8.11), for $n \geqslant 1$ we obtain

$$|\pi z(t;\ \varepsilon)| \geqslant r(t-\beta_n+\theta) - \varepsilon(ce^{K\theta} + c(t-\beta_n)) - \tag{8.12}$$
$$Dc(t-\beta_n) = \rho(t-\beta_n; \varepsilon), \quad t \in I_n$$

Here

$$\rho(s;\ \varepsilon) = r(\theta+s) - Dc(s) - \varepsilon(c(s) + ce^{K\theta}) \geqslant \rho_*(s) \tag{8.13}$$
$$s \in [0,\ \theta];\ \rho_*(s) = \rho(s;\ \varepsilon_*)$$

For $n \geqslant 0$ we have as well the representation (see (8.7) for the notation)

$$\pi z(t;\ \varepsilon) = \pi z_n(t;\ 0) + x_n(t), \quad t \in I_n \tag{8.14}$$

Since by virtue of Conditions 2 and 4 and also of inequality (8.9)

$$|\pi z_n(t;\ 0)| \geqslant r(t-\beta_n); \quad |x_n(t)| \leqslant \varepsilon c(t-\beta_n), \quad t \in I_n \tag{8.15}$$

from (8.14) we have (with $p_*(s) \equiv r(s) - \varepsilon_* c(s),\ 0 \leqslant s \leqslant \theta$)

$$|\pi z(t;\ \varepsilon)| \geqslant r(t-\beta_n) - \varepsilon c(t-\beta_n) \geqslant p_*(t-\beta_n), \quad t \in I_n \tag{8.16}$$

By virtue of the definition of $\varepsilon(t_*,\ z_*)$, from formula (8.16) we have

$$|\pi z(t;\ \varepsilon)| \geqslant r_* - \varepsilon_* c(\theta) \geqslant r_* / 2 \tag{8.17}$$

on the interval $t \in [t_* + \tau_*,\ \beta_1]$. So that in correspondence with (8.5)

$$|\pi z(t;\ \varepsilon)| \geqslant {}^1/_2 \min\ \{r_*,\ |\pi z_*|\} > 0, \quad t \in I_0$$

Setting $\mu_*(s) = \max\ \{\rho_*(s),\ p_*(s)\}$, from (8.12), (8.13) and (8.16) we have

$$|\pi z(t;\ \varepsilon)| \geqslant \mu_*(t - \beta_n), \quad t \in I_n, \quad n = 1, 2, \ldots \tag{8.18}$$

Let us show that $\mu_*(s) > 0$ on $[0,\ \theta]$. In fact (see (8.17)), $p_*(s) \geqslant r_* / 2 > 0$, $\tau_* \leqslant s \leqslant \theta$. By virtue of the definition of τ_* (see (8.4)) we have

$$\rho_*(s) \geqslant r_0 - Dc(\tau_*) - \varepsilon_* c(1 + e^{K\theta}) \geqslant r_0 / 2 - \varepsilon_*\ (e^{2K\theta} - 1) / K$$

on the interval $[0,\ \tau_*]$. Hence, according to the definition of ε_* we have $\rho_*(s) \geqslant r_0 / 4 > 0$, $s \in [0,\ \tau_*]$. The positiveness of $\mu_*(s)$ is proved.

Since $\mu_*(s)$ is continuous, we have that $l_* = \min\limits_{s \in [0,\ \theta]} \mu_*(s) > 0$, so that formulas (8.16) and (8.18) guarantee l-escape in problem (1.1) for $\varepsilon \in [0,\ \varepsilon_*]$ and $z(t_*;\ \varepsilon) = z_*$.

REFERENCES

1. Pontriagin, L. S., A Linear Differential Game of Escape. Tr. Matem. Inst. im. V. A. Steklov, Vol. 112, 1971.
2. Filippov, A. F., Differential equations with a discontinuous right-hand side. Matem. Sb., Vol. 51, № 1, 1960.
3. Mishchenko, E. F. and Satimov, N., Problem of contact evasion in differential games with nonlinear equations. Differents. Uravnen., Vol. 9, № 10, 1973.
4. Pshenichnyi, B. N. and Chirkii, A. A., Problem of contact evasion in differential games. (English translation), Pergamon Press, Zh. Vychisl. Mat. mat. Fiz., Vol. 14, № 6, 1974.
5. Gusiatnikov, P. B., Escape of nonlinear objects of different types with integral constraints on the control. PMM Vol. 39, № 1, 1975.
6. Gusiatnikov, P. B., On the l-evasion of contact in a linear differential game. PMM Vol. 38, № 3, 1974.
7. Hartman, P., Ordinary Differential Equations. New York, J. Wiley and Sons, Inc., 1964.
8. Nikol'skii, M. S., On an escape method. Dokl. Akad. Nauk SSSR, Vol. 214, № 2, 1974.

Translated by N. H. C.

UDC 531.31

NONCANONICAL INVARIANTS OF HAMILTONIAN SYSTEMS

PMM Vol. 40, № 1, 1976, pp. 38-43

L. M. MARKHASHOV
(Moscow)

(Received December 23, 1974)

We explain the character of simplifications which can be carried out in the Hamiltonian function of a nonresonant system using the formal, noncanonical transformations. We show the symmetries of such systems, which are not generated by their first integrals. Using a Hamiltonian system with two degrees of freedom we show that the noncanonical transformations retaining its normal form but with